Functional determinants for general Sturm–Liouville problems

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Abstract
Simple and analytically tractable expressions for functional determinants are
known to exist for many cases of interest. We extend the range of situations
for which these hold to cover systems of self-adjoint operators of the Sturm–
Liouville type with arbitrary linear boundary conditions. The results hold
whether or not the operators have negative eigenvalues. The physically
important case of functional determinants of operators with a zero mode, but
where that mode has been extracted, is studied in detail for the same range of
situations as when no zero mode exists. The method of proof uses the properties
of generalized zeta-functions. The general form of the final results is the same
for the entire range of problems considered.

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1. Introduction

This paper is concerned with the rather elegant, and surprisingly simple, expressions that exist
for the functional determinants of certain types of differential operators. In an earlier paper [1],
we introduced a new method for deriving these expressions for operators of a relatively simple
kind, which only used elementary ideas from complex analysis and the theory of differential
equations. Here we extend the class of problems which may be analysed using this technique.
Although the discussion necessarily becomes more technical, the essential points remain the
same, and we are able to derive the desired results without the need for any very sophisticated
machinery.

The derivation of formulae of this kind is a topic which has been investigated by numerous
authors in the past. In our earlier paper [1], we gave a brief history of the subject. Essentially,
most of the early results were obtained by theoretical physicists who were typically interested
in the expressions obtained when carrying out Gaussian functional integrals [2–4]. These
results were then extended and elaborated in a number of ways [5–19]. However, many of
these latter treatments were quite abstract, and also did not deal with the case where the

operator has a zero eigenvalue (a 'zero mode'). This situation is quite commonly encountered in real problems in theoretical physics, since in many cases a continuous symmetry in the problem is broken, and a zero mode is generated by Goldstone's theorem [20]. Although there has been some work carried out to determine the form of functional determinants with zero modes excluded [21–23], the methods that were used involved the use of a regularization procedure which could have produced results which were not independent of the scheme adopted.

These were the motivations for our approach described in [1]. The method used a generalized zeta-function [24–26] to calculate the functional determinants, but the analysis involved only methods which are familiar to theoretical physicists. It also covered the physically interesting situation where operators had zero modes which were excluded from the evaluation of the functional determinants. The method was described for simple operators of the type $-\mathrm{d}^2/\mathrm{d}x^2 + R(x)$, but for general linear boundary conditions. In the present paper we extend this treatment in several ways. Firstly, we derive the results for the general Sturm–Liouville operator $-\mathrm{d}/\mathrm{d}x(P(x)\,\mathrm{d}/\mathrm{d}x) + R(x)$. Secondly, we allow for the fact that operators will, in general, have negative eigenvalues. Thirdly, we generalize the entire formalism to systems of second-order operators. In all cases we derive the results for functional determinants of operators which do not have a zero mode, and for those which do, but where it has been extracted.

The outline of the paper is as follows. In section 2 we discuss the formalism for the general Sturm–Liouville operator, modifying our previous treatment to cover the case of arbitrary $P(x) > 0$ and operators with negative eigenvalues. We restrict ourselves to operators with no zero mode; this case is discussed in section 3. In section 4 it is shown how the results of these two sections carry over to systems of $r > 1$ degrees of freedom. We conclude in section 5 with a summary of the results of the paper and suggestions for future work. There are three appendices. In appendix A we discuss the conditions which have to be imposed so that the operator is self-adjoint and give details of some technical calculations that are required in the development of the theory. In appendix B some results on the asymptotic form of solutions of differential equations, which are used in the main text, are derived. In appendix C some of the more technical aspects of dealing with zero modes in systems of differential equations are presented.

2. One component

In this section we will describe our approach in the context of operators of the form

$$L_j = -\frac{\mathrm{d}}{\mathrm{d}x}\left(P_j(x)\frac{\mathrm{d}}{\mathrm{d}x}\right) + R_j(x) \tag{1}$$

on the interval $I = [0, 1]$. The structure displayed in (1) is the most general that is possible for a self-adjoint second-order differential operator of the Sturm–Liouville type. The functions $P_j(x)$ and $R_j(x)$ are assumed to be continuous on the interval I. In addition, we assume that the metric, $P_j(x)$, is positive throughout the interval under consideration. The index j takes on only two values: the operator L_1 is the real focus of interest, but in order to control divergences in $\det L_1$, we actually consider the ratio $\det L_1 / \det L_2$, where L_2 is appropriately chosen. Typically L_2 will be taken to be 'simple' in a sense that it can act as a reference with which $\det L_1$ can be compared.

The eigenproblem corresponding to (1) is

$$L_j u_{j,\sqrt{\lambda}}(x) = \lambda u_{j,\sqrt{\lambda}}(x). \tag{2}$$

Note the symmetry $u_{j,\sqrt{\lambda}} = u_{j,-\sqrt{\lambda}}$. It is convenient to go over to a first-order formalism and in order to have the most natural formulation we define a new function $v_{j,\sqrt{\lambda}}(x) \equiv P_j(x)u'_{j,\sqrt{\lambda}}(x)$. Then from (1) we have that

$$\frac{d}{dx}\begin{pmatrix} u_{j,\sqrt{\lambda}}(x) \\ v_{j,\sqrt{\lambda}}(x) \end{pmatrix} = \begin{pmatrix} 0 & P_j^{-1}(x) \\ R_j(x) - \lambda & 0 \end{pmatrix}\begin{pmatrix} u_{j,\sqrt{\lambda}}(x) \\ v_{j,\sqrt{\lambda}}(x) \end{pmatrix}.$$ (3)

We will adopt the notation

$$\mathbf{u}_{j,\sqrt{\lambda}}(x) = \begin{pmatrix} u_{j,\sqrt{\lambda}}(x) \\ v_{j,\sqrt{\lambda}}(x) \end{pmatrix} \qquad D_{j,\lambda} = \begin{pmatrix} 0 & P_j^{-1}(x) \\ R_j(x) - \lambda & 0 \end{pmatrix}$$ (4)

in which case (3) may be written as

$$\frac{d\mathbf{u}_{j,\sqrt{\lambda}}(x)}{dx} = D_{j,\lambda}(x)\mathbf{u}_{j,\sqrt{\lambda}}(x).$$ (5)

It is useful at this stage to introduce two unique, independent solutions of the differential equation (2). The solutions are made unique by specifying the 'initial conditions', that is, the value of the solutions and their derivatives at $x = 0$. Denoting these two solutions by $u^{(1)}_{j,\sqrt{\lambda}}(x)$ and $u^{(2)}_{j,\sqrt{\lambda}}(x)$, the most general solution of (2) may then be expressed as

$$\begin{pmatrix} u_{j,\sqrt{\lambda}}(x) \\ v_{j,\sqrt{\lambda}}(x) \end{pmatrix} = \alpha\begin{pmatrix} u^{(1)}_{j,\sqrt{\lambda}}(x) \\ v^{(1)}_{j,\sqrt{\lambda}}(x) \end{pmatrix} + \beta\begin{pmatrix} u^{(2)}_{j,\sqrt{\lambda}}(x) \\ v^{(2)}_{j,\sqrt{\lambda}}(x) \end{pmatrix} = \begin{pmatrix} u^{(1)}_{j,\sqrt{\lambda}}(x) & u^{(2)}_{j,\sqrt{\lambda}}(x) \\ v^{(1)}_{j,\sqrt{\lambda}}(x) & v^{(2)}_{j,\sqrt{\lambda}}(x) \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix}.$$ (6)

We now define the two matrices

$$E_{j,\sqrt{\lambda}}(x) = \begin{pmatrix} u^{(1)}_{j,\sqrt{\lambda}}(x) & u^{(2)}_{j,\sqrt{\lambda}}(x) \\ v^{(1)}_{j,\sqrt{\lambda}}(x) & v^{(2)}_{j,\sqrt{\lambda}}(x) \end{pmatrix} \qquad H_{j,\sqrt{\lambda}}(x) = \begin{pmatrix} u^{(1)}_{j,\sqrt{\lambda}}(x) & u^{(2)}_{j,\sqrt{\lambda}}(x) \\ u^{(1)'}_{j,\sqrt{\lambda}}(x) & u^{(2)'}_{j,\sqrt{\lambda}}(x) \end{pmatrix}$$ (7)

which are related by

$$E_{j,\sqrt{\lambda}}(x) = \begin{pmatrix} 1 & 0 \\ 0 & P_j(x) \end{pmatrix} H_{j,\sqrt{\lambda}}(x).$$ (8)

It follows that $\det E_{j,\sqrt{\lambda}}(x) = P_j(x)\det H_{j,\sqrt{\lambda}}(x)$. Since $u^{(1)}_{j,\sqrt{\lambda}}(x)$ and $u^{(2)}_{j,\sqrt{\lambda}}(x)$ are independent solutions, $\det H_{j,\sqrt{\lambda}} \neq 0$, and therefore $\det E_{j,\sqrt{\lambda}} \neq 0$ because $P_j(x) > 0\,\forall x$. Furthermore, because $\det E_{j,\sqrt{\lambda}}(x)$ is the Wronski determinant for the differential operator (4), we see that $\det E_{j,\sqrt{\lambda}}(x)$ is independent of x. A convenient choice for the set of initial conditions is $E_{j,\sqrt{\lambda}}(0) = I_2$ (here, and throughout the paper, I_m is the $m \times m$ unit matrix). From this it follows that $\det E_{j,\sqrt{\lambda}}(x) = 1\,\forall x \in [0, 1]$. Also with this choice for the initial conditions on $u^{(1)}_{j,\sqrt{\lambda}}(x)$ and $u^{(2)}_{j,\sqrt{\lambda}}(x)$, it follows from (6) that $\alpha = u_{j,\sqrt{\lambda}}(0)$ and $\beta = v_{j,\sqrt{\lambda}}(0)$, that is,

$$u_{j,\sqrt{\lambda}}(x) = u_{j,\sqrt{\lambda}}(0)u^{(1)}_{j,\sqrt{\lambda}}(x) + v_{j,\sqrt{\lambda}}(0)u^{(2)}_{j,\sqrt{\lambda}}(x)$$ (9)

or in terms of the first-order formalism,

$$\begin{pmatrix} u_{j,\sqrt{\lambda}}(x) \\ v_{j,\sqrt{\lambda}}(x) \end{pmatrix} = E_{j,\sqrt{\lambda}}(x)\begin{pmatrix} u_{j,\sqrt{\lambda}}(0) \\ v_{j,\sqrt{\lambda}}(0) \end{pmatrix}.$$ (10)

So far no mention has been made of the boundary conditions on (2). These take the form of two conditions on the set $\{u_{j,\sqrt{\lambda}}(0), u'_{j,\sqrt{\lambda}}(0), u_{j,\sqrt{\lambda}}(1), u'_{j,\sqrt{\lambda}}(1)\}$. These can be converted

into conditions on $\mathbf{u}_{j,\sqrt{\lambda}}$ at the boundaries, and for the case of linear boundary conditions

$$M \begin{pmatrix} u_{j,\sqrt{\lambda}}(0) \\ v_{j,\sqrt{\lambda}}(0) \end{pmatrix} + N \begin{pmatrix} u_{j,\sqrt{\lambda}}(1) \\ v_{j,\sqrt{\lambda}}(1) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{11}$$

where M and N are 2×2 matrices whose entries characterize the nature of the boundary conditions. Using (10) these boundary conditions may be written as

$$[M + N E_{j,\sqrt{\lambda}}(1)] \begin{pmatrix} u_{j,\sqrt{\lambda}}(0) \\ v_{j,\sqrt{\lambda}}(0) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{12}$$

and so the condition on λ for eigenfunctions to exist is

$$\det[M + N E_{j,\sqrt{\lambda}}(1)] = 0. \tag{13}$$

The equations (11) are the most general linear boundary conditions. They fall naturally into two classes:

(i) $\det M = 0$, $\det N = 0$. In this case we can show that the matrices M and N may be chosen to be of the form,

$$M = \begin{pmatrix} A & B \\ 0 & 0 \end{pmatrix} \qquad N = \begin{pmatrix} 0 & 0 \\ C & D \end{pmatrix} \tag{14}$$

that is, the boundary conditions are of the Robin type: $A u_{j,\sqrt{\lambda}}(0) + B v_{j,\sqrt{\lambda}}(0) = 0$ and $C u_{j,\sqrt{\lambda}}(1) + D v_{j,\sqrt{\lambda}}(1) = 0$.

First, let us prove that $\det M = 0$ and $\det N = 0$ implies that $M \mathbf{u}_{j,\sqrt{\lambda}}(0) = 0$ and $N \mathbf{u}_{j,\sqrt{\lambda}}(1) = 0$. To see this, define $\mathbf{u}'_{j,\sqrt{\lambda}}(x) = Q^{-1} \mathbf{u}_{j,\sqrt{\lambda}}(x)$ and multiply (11) by P, where P and Q are arbitrary non-singular matrices. Then defining $M' = PMQ$ and $N' = PNQ$, we obtain the same boundary conditions but in the primed system. However, since M has rank 1, we may choose P and Q in such a way that $M' = \text{diag}(1, 0)$ or $\text{diag}(0, 1)$. Furthermore, since N' has zero determinant it must have one of the following four forms:

$$A_1 = \begin{pmatrix} a & b \\ ka & kb \end{pmatrix} \qquad A_2 = \begin{pmatrix} ka & kb \\ a & b \end{pmatrix}$$

$$A_3 = \begin{pmatrix} a & ka \\ b & kb \end{pmatrix} \qquad A_4 = \begin{pmatrix} ka & a \\ kb & b \end{pmatrix}.$$

Writing out the boundary conditions explicitly when $M' = \text{diag}(1, 0)$ and N' has each of these four forms, we find that for all cases where there are two independent conditions, $u'_{j,\sqrt{\lambda}}(0) = 0$, that is, $M' \mathbf{u}'_{j,\sqrt{\lambda}}(0) = 0$. Similarly, if $M' = \text{diag}(0, 1)$, we find that for all four possible forms of N', all valid boundary conditions lead to $v'_{j,\sqrt{\lambda}}(0) = 0$, which for this choice of M' once again gives $M' \mathbf{u}'_{j,\sqrt{\lambda}}(0) = 0$. Returning to the unprimed system this implies that $M \mathbf{u}_{j,\sqrt{\lambda}}(0) = 0$ and so from (11), $N \mathbf{u}_{j,\sqrt{\lambda}}(1) = 0$, as required.

We may now use the fact that M and N separately must have one of the forms $A_1, \ldots, A_4$. In each case there is only one independent relation of the Robin type. This may be written in the language of M and N matrices by adopting the forms (14).

(ii) $\det N \neq 0$. In appendix B we show that this implies that $\det M \neq 0$. Then, from (11),

$$\begin{pmatrix} u_{j,\sqrt{\lambda}}(0) \\ v_{j,\sqrt{\lambda}}(0) \end{pmatrix} = -M^{-1} N \begin{pmatrix} u_{j,\sqrt{\lambda}}(1) \\ v_{j,\sqrt{\lambda}}(1) \end{pmatrix}.$$
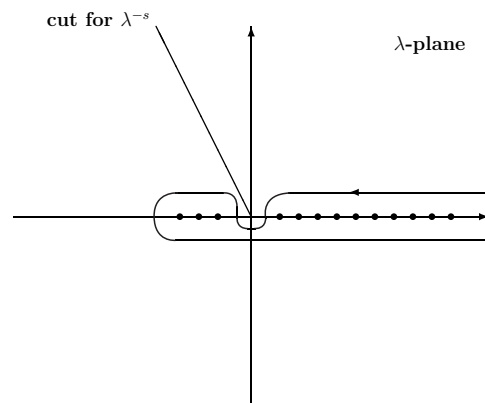
Figure 1. Contour γ in the complex plane.

Since N is not null, neither is $M^{-1}N$, and so either $u_{j,\sqrt{\lambda}}(0)$ or $v_{j,\sqrt{\lambda}}(0)$ depend on the boundary conditions at $x = 1$. Boundary conditions such as these are called two-point boundary conditions, or non-separated boundary conditions, in contrast to the one-point or separated boundary conditions described by (14).

After this short review of the background, we are now in a position to describe our method for obtaining the basic formula for $\det L_1 / \det L_2$. The starting point is the observation from (13) that the function $\det[M + NE_{j,\sqrt{\lambda}}(1)]$ has zeros at values λ which are eigenvalues of L_j, as given by (2). An alternative statement is that the logarithmic derivative of $\det[M + NE_{j,\sqrt{\lambda}}(1)]$ has a simple pole with unit residue at these values of λ. This allows us to define the zeta function of L_j by

$$\zeta_{L_j}(s) = \frac{1}{2\pi\mathrm{i}} \int_\gamma \mathrm{d}\lambda\, \lambda^{-s} \frac{\mathrm{d}}{\mathrm{d}\lambda} \ln \det[M + NE_{j,\sqrt{\lambda}}(1)] \tag{15}$$

where the contour γ is counterclockwise and encloses all eigenvalues as shown in figure 1.

As given, the representation is valid for $\mathrm{Re}\,s > 1/2$. In this section we assume that there are no zero modes, but note that we allow for negative eigenvalues. In order to avoid the negative eigenvalues lying on the cut of the complex square root, we define the branch cut to be at an angle θ to the positive real axis. For most applications it is the ratio of determinants of two operators that naturally occurs. This is found by analysing

$$\zeta_{L_1}(s) - \zeta_{L_2}(s) = \frac{1}{2\pi\mathrm{i}} \int_\gamma \mathrm{d}\lambda\, \lambda^{-s} \frac{\mathrm{d}}{\mathrm{d}\lambda} \ln \frac{\det[M + NE_{1,\sqrt{\lambda}}(1)]}{\det[M + NE_{2,\sqrt{\lambda}}(1)]}. \tag{16}$$

The first idea is to deform the contour such that it encloses the branch cut of λ^{-s}. In order to see in which range of s-values this is possible, let us consider the large-$\mathrm{Im}\,\sqrt{\lambda}$ behaviour of the integrand.

As shown in appendix B, we have for $P_1(x) = P_2(x)$ as $\mathrm{Im}\,\sqrt{\lambda} \to \pm\infty$ the behaviour

$$\frac{\mathrm{d}}{\mathrm{d}\lambda} \ln \frac{\det[M + NE_{1,\sqrt{\lambda}}(1)]}{\det[M + NE_{2,\sqrt{\lambda}}(1)]} = \mathcal{O}\left(\frac{1}{\lambda^{3/2}}\right). \tag{17}$$

So for $-1/2 < \mathrm{Re}\,s < 1$ we can shift the contour such as to enclose the cut and ultimately we can shrink it to the cut. Taking due care of the definition of the complex root near the cut, we find for the upper part

$$\zeta^u_{L_1}(s) - \zeta^u_{L_2}(s) = -\frac{1}{2\pi\mathrm{i}}\, \mathrm{e}^{-\mathrm{i}s\theta} \int_0^\infty \mathrm{d}\lambda\, \lambda^{-s} \frac{\mathrm{d}}{\mathrm{d}\lambda} \ln \frac{\det[M + NE_{1,\mathrm{e}^{\mathrm{i}\theta/2}\sqrt{\lambda}}(1)]}{\det[M + NE_{2,\mathrm{e}^{\mathrm{i}\theta/2}\sqrt{\lambda}}(1)]}$$

whereas for the lower part we have

$$\zeta_{L_1}^l(s) - \zeta_{L_2}^l(s) = \frac{1}{2\pi i} e^{is(2\pi-\theta)} \int_0^\infty d\lambda \, \lambda^{-s} \frac{d}{d\lambda} \ln \frac{\det[M + NE_{1,e^{-i(\pi-\theta/2)}\sqrt{\lambda}}(1)]}{\det[M + NE_{2,e^{-i(\pi-\theta/2)}\sqrt{\lambda}}(1)]}.$$

Using the symmetry $E_{j,e^{i\theta/2}\sqrt{\lambda}}(1) = E_{j,e^{-i(\pi-\theta/2)}\sqrt{\lambda}}(1)$, these contributions add up to yield

$$\zeta_{L_1}(s) - \zeta_{L_2}(s) = \frac{1}{2\pi i} (e^{is(2\pi-\theta)} - e^{-is\theta}) \int_0^\infty d\lambda \, \lambda^{-s} \frac{d}{d\lambda} \ln \frac{\det[M + NE_{1,e^{i\theta/2}\sqrt{\lambda}}(1)]}{\det[M + NE_{2,e^{i\theta/2}\sqrt{\lambda}}(1)]}$$

$$= e^{is(\pi-\theta)} \frac{\sin(\pi s)}{\pi} \int_0^\infty d\lambda \, \lambda^{-s} \frac{d}{d\lambda} \ln \frac{\det[M + NE_{1,e^{i\theta/2}\sqrt{\lambda}}(1)]}{\det[M + NE_{2,e^{i\theta/2}\sqrt{\lambda}}(1)]}. \tag{18}$$

For $\theta = \pi$ and $P_j(x) = 1$ this reduces to the result of our previous paper [1]. This type of result is now perfectly suited for the evaluation of the determinant quotient. The prefactor disappears at $s = 0$ and so

$$\zeta_{L_1}'(0) - \zeta_{L_2}'(0) = -\ln \frac{\det[M + NE_{1,0}(1)]}{\det[M + NE_{2,0}(1)]}. \tag{19}$$

In particular, we note that the answer does not depend on the angle θ. Simplifying notation we define

$$y_j(x) = \lim_{\sqrt{\lambda}\to 0} u_{j,\sqrt{\lambda}}(x) \qquad y_j^{(a)}(x) = \lim_{\sqrt{\lambda}\to 0} u_{j,\sqrt{\lambda}}^{(a)}(x) (a = 1, 2) \qquad Y_j(x) = \lim_{\sqrt{\lambda}\to 0} E_{j,\sqrt{\lambda}}(x). \tag{20}$$

We will refer to $y_j(x)$ and $y_j^{(a)}(x)$ as homogeneous solutions since they all satisfy the equation $L_j y_j(x) = 0$. Then the result is from (19)

$$\frac{\det L_1}{\det L_2} = \frac{\det[M + NY_1(1)]}{\det[M + NY_2(1)]}. \tag{21}$$

This is formally identical to the result we obtained when $P_j(x) = 1$ and all of the eigenvalues were positive [1]. This shows that this simple result is obtained even with the added complications of non trivial metrics and negative eigenvalues, as long as L_2 is chosen so that $P_2(x) = P_1(x)$.

3. Determinants with zero modes extracted

In this section we discuss the evaluation of determinants of operators which have a zero eigenvalue and where this eigenvalue has been extracted in the definition of the determinant. We shall indicate this exclusion with a prime: thus $\det' L$ will denote the determinant of the operator L with the zero mode extracted. Clearly the method used in the last section to derive the formula for the ratio of determinants runs into difficulty when evaluating such determinants. Even if the contour γ is chosen to only surround the non-zero values of λ, the deformation of this contour to the branch cut will encounter the pole at the origin. Rather than dealing directly with this extra singularity, we can look for a function which vanishes at all the non-zero values of λ but not at the zero eigenvalue. This function can then be used as the basis of the definition of a (modified) zeta-function, from which $\det' L$ can be calculated. As we will show in this section, the quantity $f_{1,\sqrt{\lambda}} \equiv (-1/\lambda) \det(M + NE_{1,\sqrt{\lambda}}(1))$ has these properties: it clearly vanishes at all the required non-zero values of λ by (13) and we will show that $f_{1,0} \neq 0$. Since we will assume that the 'normalizing' operator L_2 has no zero modes, j will be set equal to 1 throughout any discussion involving zero modes.

The first step in the derivation is relevant even if there is no zero mode. It consists of demanding that the solution $u_{j,\sqrt{\lambda}}(x) = \alpha u_{j,\sqrt{\lambda}}^{(1)}(x) + \beta u_{j,\sqrt{\lambda}}^{(2)}(x)$ satisfies one of the boundary

conditions. We may choose either one of the conditions to be satisfied, but in general it will fix the functional form of $u_{j,\sqrt{\lambda}}(x)$ by determining the ratio of α to β. In the special cases of Dirichlet and Neumann boundary conditions it will result in α and β, respectively, being set equal to zero. Although the normalization of $u_{j,\sqrt{\lambda}}(x)$ is obviously left undetermined, we shall now show that a *suitable choice of normalization results in a significant simplification of the analysis*.

To see this let us write out the boundary conditions (11) in full:

$$m_{11}u_{j,\sqrt{\lambda}}(0) + m_{12}v_{j,\sqrt{\lambda}}(0) + n_{11}u_{j,\sqrt{\lambda}}(1) + n_{12}v_{j,\sqrt{\lambda}}(1) = 0 \tag{22}$$

$$m_{21}u_{j,\sqrt{\lambda}}(0) + m_{22}v_{j,\sqrt{\lambda}}(0) + n_{21}u_{j,\sqrt{\lambda}}(1) + n_{22}v_{j,\sqrt{\lambda}}(1) = 0. \tag{23}$$

Now consider the explicit form of the matrix $M + NE_{j,\sqrt{\lambda}}(1)$:

$$\begin{pmatrix} m_{11} + n_{11}u^{(1)}_{j,\sqrt{\lambda}}(1) + n_{12}v^{(1)}_{j,\sqrt{\lambda}}(1) & m_{12} + n_{11}u^{(2)}_{j,\sqrt{\lambda}}(1) + n_{12}v^{(2)}_{j,\sqrt{\lambda}}(1) \\ m_{21} + n_{21}u^{(1)}_{j,\sqrt{\lambda}}(1) + n_{22}v^{(1)}_{j,\sqrt{\lambda}}(1) & m_{22} + n_{21}u^{(2)}_{j,\sqrt{\lambda}}(1) + n_{22}v^{(2)}_{j,\sqrt{\lambda}}(1) \end{pmatrix} \tag{24}$$

where we have used the definition of $E_{j,\sqrt{\lambda}}$ given in (7). If we add β/α times column 2 to column 1 of (24) we get a second matrix whose first column is just α^{-1} times the boundary conditions given in (22) and (23) (remembering that $E_{j,\sqrt{\lambda}}(0) = I_2$). So suppose we ask that only the boundary condition (22) is satisfied. Since the determinants of both these matrices are equal, it follows that

$$\det(M + NE_{j,\sqrt{\lambda}}(1)) = \alpha^{-1}\{m_{21}u_{j,\sqrt{\lambda}}(0) + m_{22}v_{j,\sqrt{\lambda}}(0) + n_{21}u_{j,\sqrt{\lambda}}(1) + n_{22}v_{j,\sqrt{\lambda}}(1)\}$$
$$\times (-1)\left(m_{12} + n_{11}u^{(2)}_{j,\sqrt{\lambda}}(1) + n_{12}v^{(2)}_{j,\sqrt{\lambda}}(1)\right).$$

Therefore, if we make the choice

$$\alpha = -\left(m_{12} + n_{11}u^{(2)}_{j,\sqrt{\lambda}}(1) + n_{12}v^{(2)}_{j,\sqrt{\lambda}}(1)\right) \tag{25}$$

then

$$\det(M + NE_{j,\sqrt{\lambda}}(1)) = m_{21}u_{j,\sqrt{\lambda}}(0) + m_{22}v_{j,\sqrt{\lambda}}(0) + n_{21}u_{j,\sqrt{\lambda}}(1) + n_{22}v_{j,\sqrt{\lambda}}(1). \tag{26}$$

Similarly if we add α/β of column 1 to column 2 of (24) we get another matrix whose second column is just β^{-1} times the boundary conditions given in (22) and (23). Again choosing the first boundary condition to be satisfied, and also now asking that (26) holds, then we determine β to be given by

$$\beta = \left(m_{11} + n_{11}u^{(1)}_{j,\sqrt{\lambda}}(1) + n_{12}v^{(1)}_{j,\sqrt{\lambda}}(1)\right). \tag{27}$$

So in summary, we have shown that if we take a solution of (2) of the form

$$u_{j,\sqrt{\lambda}}(x) = -\left(m_{12} + n_{11}u^{(2)}_{j,\sqrt{\lambda}}(1) + n_{12}v^{(2)}_{j,\sqrt{\lambda}}(1)\right)u^{(1)}_{j,\sqrt{\lambda}}(x)$$
$$+ \left(m_{11} + n_{11}u^{(1)}_{j,\sqrt{\lambda}}(1) + n_{12}v^{(1)}_{j,\sqrt{\lambda}}(1)\right)u^{(2)}_{j,\sqrt{\lambda}}(x) \tag{28}$$

then

$$M\begin{pmatrix} u_{j,\sqrt{\lambda}}(0) \\ v_{j,\sqrt{\lambda}}(0) \end{pmatrix} + N\begin{pmatrix} u_{j,\sqrt{\lambda}}(1) \\ v_{j,\sqrt{\lambda}}(1) \end{pmatrix} = \begin{pmatrix} 0 \\ \det(M + NE_{j,\sqrt{\lambda}}(1)) \end{pmatrix}. \tag{29}$$

That is, if $u_{j,\sqrt{\lambda}}(x)$ is chosen to satisfy only one boundary condition (in this case the first), and its normalization is chosen appropriately, then $\det(M + NE_{j,\sqrt{\lambda}}(1))$ will be directly proportional to the expression on the left-hand side of the boundary condition (23), with a

constant of proportionality equal to unity. As far as we are concerned in this paper, the relation (29) has two important consequences:

(i) If there is no zero mode in the problem, we can simply take the limit $\lambda \to 0$ in the above formulae and get a simplified, and more explicit, expression for the result (21). To do this we choose the particular solution of the homogeneous equation $L_j y_j(x) = 0$ to be

$$
\begin{aligned}
y_j(x) = &-\left(m_{12} + n_{11} y_j^{(2)}(1) + n_{12} P_j(1) y_j^{(2)'}(1)\right) y_j^{(1)}(x) \\
&+ \left(m_{11} + n_{11} y_j^{(1)}(1) + n_{12} P_j(1) y_j^{(1)'}(1)\right) y_j^{(2)}(x).
\end{aligned}
\tag{30}
$$

In terms of this particular solution, the $\lambda \to 0$ limit of (26) may be used to write the result (21) as

$$
\frac{\det L_1}{\det L_2} = \frac{m_{21} y_1(0) + m_{22} P_1(0) y_1'(0) + n_{21} y_1(1) + n_{22} P_1(1) y_1'(1)}{m_{21} y_2(0) + m_{22} P_2(0) y_2'(0) + n_{21} y_2(1) + n_{22} P_2(1) y_2'(1)}.
\tag{31}
$$

(ii) If there is a zero mode, instead of taking the limit $\lambda \to 0$, we use (26) as the source of the two relationships we need to show, namely that $\det(M + N E_{1,\sqrt{\lambda}}(1)) \sim \lambda$ for small $|\lambda|$, and in particular that $f_{1,\sqrt{\lambda}}$ defined earlier, satisfies $f_{1,0} \neq 0$. We now discuss in more detail how this is carried out.

Let us begin by defining the Hilbert space product of $u_{1,\sqrt{\lambda}}(x)$ and $u_{1,0}(x)$ on $\mathcal{L}^2(I)$ by

$$
\langle u_{1,0} | u_{1,\sqrt{\lambda}} \rangle = \int_0^1 \mathrm{d}x \, u_{1,0}(x)^* u_{1,\sqrt{\lambda}}(x)
\tag{32}
$$

where * denotes complex conjugation. So multiplying (2) by $u_{1,0}(x)^*$ and integrating gives

$$
\int_0^1 \mathrm{d}x \, u_{1,0}(x)^* L_1 u_{1,\sqrt{\lambda}}(x) = \lambda \langle u_{1,0} | u_{1,\sqrt{\lambda}} \rangle.
$$

By partial integration we get boundary terms plus $L_1 u_{1,0}(x)^*$. This latter term is zero, so therefore

$$
[u_{1,\sqrt{\lambda}}(x) v_{1,0}(x)^* - u_{1,0}(x)^* v_{1,\sqrt{\lambda}}(x)]_0^1 = \lambda \langle u_{1,0} | u_{1,\sqrt{\lambda}} \rangle.
\tag{33}
$$

We can now use (29) to solve for two members of the set $\{u_{1,\sqrt{\lambda}}(0), v_{1,\sqrt{\lambda}}(0), u_{1,\sqrt{\lambda}}(1), v_{1,\sqrt{\lambda}}(1)\}$ in terms of the other two and $\det(M + N E_{1,\sqrt{\lambda}}(1))$. An exactly analogous procedure is carried out on the set $\{u_{1,0}(0)^*, v_{1,0}(0)^*, u_{1,0}(1)^*, v_{1,0}(1)^*\}$, but in this case $u_{1,0}(x)$ satisfies both of the boundary conditions, and so (11), rather than (29), should be used. This procedure is discussed in more detail in appendix A, where it is shown that substituting the expressions for these four quantities into the left-hand side of equation (33) shows that it is directly proportional to $\det(M + N E_{1,\sqrt{\lambda}}(1))$. The constant of proportionality (denoted by $\mathcal{B}^{-1}$) is independent of λ, and only depends on the nature of the boundary conditions and on the $\lambda = 0$ solution $\mathbf{u}_{1,0}(x)$ at the boundaries. Therefore we may write

$$
\det(M + N E_{1,\sqrt{\lambda}}(1)) = \mathcal{B}[u_{1,\sqrt{\lambda}}(x) v_{1,0}(x)^* - u_{1,0}(x)^* v_{1,\sqrt{\lambda}}(x)]_0^1 = \mathcal{B}\lambda \langle u_{1,0} | u_{1,\sqrt{\lambda}} \rangle.
\tag{34}
$$

It should be stressed that while $u_{1,0}(x)$ satisfies both boundary conditions, $u_{1,\sqrt{\lambda}}(x)(\lambda \neq 0)$ satisfies only one (together with a normalization condition), in other words, it has the form (28). If the other boundary condition is imposed, λ is restricted to take on values for which λ is an eigenvalue, and we see from (34) that the orthogonality condition $\langle u_{1,0} | u_{1,\sqrt{\lambda}} \rangle = 0$ holds by virtue of (13).

The constant $\mathcal{B}$ is determined in appendix A, where the related question of the conditions for the operator to be self-adjoint, is also discussed. The conclusions are

(i) If the boundary conditions are separated, so that $\det M = \det N = 0$, then if the operator is self-adjoint, M and N can always be chosen to be of the form (14), with M and N real. In this case

$$\mathcal{B} = \frac{n_{21}}{v_{1,0}(1)^*} \quad \text{if } n_{21} \neq 0 \qquad \mathcal{B} = -\frac{n_{22}}{u_{1,0}(1)^*} \quad \text{if } n_{22} \neq 0. \tag{35}$$

(ii) If the boundary conditions are non-separated, so that $\det M \neq 0$, $\det N \neq 0$, then if the operator is self-adjoint, M and N can always be chosen so that one of them is real, say $N = N_R$, and the other one a real matrix times a phase: $M = M_R \, e^{i\alpha}, 0 \leqslant \alpha < 2\pi$. It also follows that $\det M_R = \det N_R$. In this case

$$\mathcal{B} = \frac{n_{12}n_{21} - n_{11}n_{22}}{n_{11}u_{1,0}(1)^* + n_{12}v_{1,0}(1)^*}. \tag{36}$$

The function $f_{1,\sqrt{\lambda}}$ mentioned earlier can now be identified. If we define

$$f_{1,\sqrt{\lambda}} \equiv -\frac{\det(M + NE_{1,\sqrt{\lambda}}(1))}{\lambda} = -\mathcal{B}\langle u_{1,0}|u_{1,\sqrt{\lambda}}\rangle \tag{37}$$

we see that it vanishes at the required values of λ, but is non-zero when $\lambda = 0$. However, in our evaluation of the contour integral in the last section, it was also vital that the large $|\sqrt{\lambda}|$ behaviour for $j = 1$ and $j = 2$ were the same, so actually we need to replace $\det(M + NE_{1,\sqrt{\lambda}}(1))$ in the integrand of the contour integral by $(1 - \lambda)f_{1,\sqrt{\lambda}}$. This has the required properties when both $\lambda = 0$ and $\lambda \neq 0$, but in addition it behaves like $\det(M + NE_{1,\sqrt{\lambda}}(1))$ for large $|\sqrt{\lambda}|$, also as required. So in order to derive an expression for $\det' L_1/\det L_2$ we need to begin from

$$\zeta_{L_1}(s) - \zeta_{L_2}(s) = -1 + \frac{1}{2\pi i}\int_\gamma d\lambda \, \lambda^{-s}\frac{d}{d\lambda}\ln\frac{(1 - \lambda)f_{1,\sqrt{\lambda}}}{\det[M + NE_{2,\sqrt{\lambda}}(1)]} \tag{38}$$

where the contour γ encloses the point $\lambda = 1$ and the values of λ on the real axis which define the eigenvalues.

It is understood that the zero mode has been omitted from the definition of the zeta function. For definiteness we have assumed that the contour encloses $\lambda = 1$ such that the term '-1' on the right-hand side corrects for the contribution due to the factor $(1 - \lambda)$. Proceeding as before, now noting that $f_{1,e^{i\theta/2}\sqrt{\lambda}} = f_{1,e^{-i(\pi-\theta/2)}\sqrt{\lambda}}$, we obtain

$$\zeta_{L_1}(s) - \zeta_{L_2}(s) = -1 + e^{is(\pi-\theta)}\frac{\sin(\pi s)}{\pi}\int_0^\infty d\lambda \, \lambda^{-s}\frac{d}{d\lambda}\ln\frac{(1 + e^{i\theta}\lambda)f_{1,e^{i\theta/2}\sqrt{\lambda}}}{\det(M + NE_{2,e^{i\theta/2}\sqrt{\lambda}}(1))}.$$

For the derivative at $s = 0$ this means

$$\zeta'_{L_1}(0) - \zeta'_{L_2}(0) = -\ln\frac{f_{1,0}}{\det(M + NE_{2,0}(1))}.$$

Using the notation of equations (20) and (37), this may be cast into the final form

$$\frac{\det' L_1}{\det L_2} = -\frac{\mathcal{B}\langle y_1|y_1\rangle}{\det[M + NY_2(1)]}. \tag{39}$$

4. Systems of differential operators

The extension from a single differential operator of the form (1) to a system of differential equations is relatively straightforward, the main problem being one of notation. Provided that the previous sections have been read, the discussion in this section should be clear, since it parallels the case of a single operator. As an additional aid to understanding, we illustrate

new concepts which are introduced on a specific example. Some of the more cumbersome formulae which are not vital to an overall understanding of the formalism are relegated to appendix C.

We consider the system of differential operators

$$L_j = -\frac{\mathrm{d}}{\mathrm{d}x}\left(P_j(x)\frac{\mathrm{d}}{\mathrm{d}x}\right)I_r + R_j(x)$$

where $R_j(x)$ is a Hermitian $r \times r$ matrix and $j = 1, 2$ labels the two different determinants. We assume $P_j(x)$ to be scalar, which is the relevant case for most applications. The second-order problem is rewritten as a first-order problem in the standard way,

$$\frac{\mathrm{d}}{\mathrm{d}x}\begin{pmatrix} u_{j,\sqrt{\lambda}}(x) \\ v_{j,\sqrt{\lambda}}(x) \end{pmatrix} = D_{j,\lambda}\begin{pmatrix} u_{j,\sqrt{\lambda}}(x) \\ v_{j,\sqrt{\lambda}}(x) \end{pmatrix}$$

with the matrix

$$D_{j,\lambda}(x) = \begin{pmatrix} 0_{r\times r} & P_j^{-1}(x) \cdot I_r \\ R_j - \lambda \cdot I_r & 0_{r\times r} \end{pmatrix}$$

and where now $u_{j,\sqrt{\lambda}}(x)$ and $v_{j,\sqrt{\lambda}}(x)$ are r-dimensional vectors. We define, as before, the fundamental matrix as

$$E_{j,\sqrt{\lambda}}(x) = \begin{pmatrix} u_{j,\sqrt{\lambda}}^{(1)}(x) & \cdots & u_{j,\sqrt{\lambda}}^{(2r)}(x) \\ v_{j,\sqrt{\lambda}}^{(1)}(x) & \cdots & v_{j,\sqrt{\lambda}}^{(2r)}(x) \end{pmatrix}$$

with $u_{j,\sqrt{\lambda}}^{(\sigma)}(x), v_{j,\sqrt{\lambda}}^{(\sigma)}(x), \sigma = 1, \ldots, 2r$, being again r-dimensional vectors. The boundary conditions read

$$M\begin{pmatrix} u_{j,\sqrt{\lambda}}(0) \\ v_{j,\sqrt{\lambda}}(0) \end{pmatrix} + N\begin{pmatrix} u_{j,\sqrt{\lambda}}(1) \\ v_{j,\sqrt{\lambda}}(1) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{40}$$

or, alternatively

$$(M + NE_{j,\sqrt{\lambda}}(1))\begin{pmatrix} u_{j,\sqrt{\lambda}}(0) \\ v_{j,\sqrt{\lambda}}(0) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{41}$$

So the condition for the eigenvalues reads

$$\det(M + NE_{j,\sqrt{\lambda}}(1)) = 0. \tag{42}$$

In the case that $P_j(x)$ is scalar, the analysis in appendix B goes through. The only change is that the heat kernel coefficients contain a trace over the internal degrees of freedom. With this change, the asymptotic behaviour of the relevant integrand is known and for $P_1(x) = P_2(x)$ we can proceed as previously. In the absence of zero modes we find formally the same answer as before,

$$\frac{\det L_1}{\det L_2} = \frac{\det(M + NY_1(1))}{\det(M + NY_2(1))}.$$

Let us next consider the case with zero modes. In order to explain the individual steps of the general formalism that we are developing, we will illustrate each step using a specific example encountered in the study of transition rates between metastable states in superconducting rings [27, 28]. The differential operator in this problem is defined on the interval $[-l/2, l/2]$ and has the form

$$L_1 = \begin{pmatrix} -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + (1 - 2\mu^2) & (1 - \mu^2)\,\mathrm{e}^{2\mathrm{i}\mu x} \\ (1 - \mu^2)\,\mathrm{e}^{-2\mathrm{i}\mu x} & -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + (1 - 2\mu^2) \end{pmatrix} \equiv -\frac{\mathrm{d}^2}{\mathrm{d}x^2}I_2 + R_1(x). \tag{43}$$

Boundary conditions imposed are so-called twisted boundary conditions defined through

$$M = -\text{diag}(e^{i\mu l}, e^{-i\mu l}, e^{i\mu l}, e^{-i\mu l}) \qquad N = I_4.$$

We will refer back to this example at suitable stages of our procedure.

The starting point for the general formalism is as before, namely (33). If we can derive a relationship of the form

$$\det(M + N E_{1,\sqrt{\lambda}}(1)) = \mathcal{B}[u_{1,\sqrt{\lambda}}(x)v_{1,0}(x)^* - u_{1,0}(x)^*v_{1,\sqrt{\lambda}}(x)]_0^1 \tag{44}$$

where $\mathcal{B}$ is known, then from (33) we have that

$$\det(M + N E_{1,\sqrt{\lambda}}(1)) = \mathcal{B}\lambda\langle u_{1,0}|u_{1,\sqrt{\lambda}}\rangle. \tag{45}$$

This is precisely as in section 3, and allows us to identify the function $f_{1,\sqrt{\lambda}}$, defined by (37), which is to be used in the proof of the result.

So, let us return to the proof of (44). We will show that, if we appropriately normalize $u_{j,\sqrt{\lambda}}(x)$, then by imposing all of the boundary conditions but one — so that λ is not constrained to be an eigenvalue—we may write

$$M \begin{pmatrix} u_{1,\sqrt{\lambda}}(0) \\ v_{1,\sqrt{\lambda}}(0) \end{pmatrix} + N \begin{pmatrix} u_{1,\sqrt{\lambda}}(1) \\ v_{1,\sqrt{\lambda}}(1) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \cdots \\ 0 \\ \det(M + N E_{1,\sqrt{\lambda}}(1)) \end{pmatrix}. \tag{46}$$

This equation is exactly analogous to (29) in section 3, where we imposed only one out of the two boundary conditions. Here there are $2r$ boundary conditions and we will impose $2r - 1$ of them.

To obtain (46), we first write $u_{j,\sqrt{\lambda}}(x)$ as a linear combination of the $2r$ fundamental solutions $u_{j,\sqrt{\lambda}}^{(\sigma)}(x)$

$$u_{j,\sqrt{\lambda}}(x) = \sum_{\sigma=1}^{2r} \alpha^{(\sigma)} u_{j,\sqrt{\lambda}}^{(\sigma)}(x) \Rightarrow v_{j,\sqrt{\lambda}}(x) = \sum_{\sigma=1}^{2r} \alpha^{(\sigma)} v_{j,\sqrt{\lambda}}^{(\sigma)}(x) \tag{47}$$

where we have dropped the j and $\sqrt{\lambda}$ dependence from the α. Since $E_{j,\sqrt{\lambda}}(0) = I_{2r}$,

$$\alpha^{(\sigma)} = \begin{cases} u_{j,\sqrt{\lambda},\sigma}(0) & \text{if } \sigma = 1, \ldots, r \\ v_{j,\sqrt{\lambda},\sigma-r}(0) & \text{if } \sigma = r+1, \ldots, 2r \end{cases}$$

where $u_{j,\sqrt{\lambda},\sigma}(x)$ is the σth entry of the vector $u_{j,\sqrt{\lambda}}(x)$, with a similar notation for $v_{j,\sqrt{\lambda}}(x)$. Therefore using (41), but only imposing the first $2r - 1$ boundary conditions, gives

$$(M + N E_{j,\sqrt{\lambda}}(1)) \begin{pmatrix} \alpha^{(1)} \\ \alpha^{(2)} \\ \cdots \\ \alpha^{(2r-1)} \\ \alpha^{(2r)} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \cdots \\ 0 \\ * \end{pmatrix}. \tag{48}$$

First, suppose that $\det(M + NE_{j,\sqrt{\lambda}}(1)) \neq 0$. Then, multiplying (48) by $(M + NE_{j,\sqrt{\lambda}}(1))^{-1}$ yields

$$
\begin{pmatrix} \alpha^{(1)} \\ \alpha^{(2)} \\ \cdots \\ \alpha^{(2r-1)} \\ \alpha^{(2r)} \end{pmatrix} = \frac{\mathrm{adj}(M + NE_{j,\sqrt{\lambda}}(1))}{\det(M + NE_{j,\sqrt{\lambda}}(1))} \begin{pmatrix} 0 \\ 0 \\ \cdots \\ 0 \\ * \end{pmatrix}
$$

where $\mathrm{adj}(M + NE_{j,\sqrt{\lambda}}(1))$ is the adjoint of the matrix $M + NE_{j,\sqrt{\lambda}}(1)$. We see that the choice of $\det(M + NE_{j,\sqrt{\lambda}}(1))$ for $*$ is natural, since in this case the expansion coefficients have the simple form

$$
\alpha^{(\sigma)} = \mathrm{adj}(M + NE_{j,\sqrt{\lambda}}(1))_{\sigma 2r}. \tag{49}
$$

In the $r = 1$ case this simply leads to the results (25) and (27). If $\det(M + NE_{j,\sqrt{\lambda}}(1)) = 0$, then by (42) $u_{j,\sqrt{\lambda}}(x)$ is an eigenfunction which satisfies the boundary conditions, and so (46) also holds.

Altogether there are $4r$ boundary data, r data coming from each of $u_{1,\sqrt{\lambda}}(0)$, $v_{1,\sqrt{\lambda}}(0)$, $u_{1,\sqrt{\lambda}}(1)$ and $v_{1,\sqrt{\lambda}}(1)$. Equation (46) allows us to express $2r$ of the boundary data in terms of the other $2r$ data, which we call the complementary ones. Suppose that b is a vector consisting of the $2r$ boundary data that we wish to express by the complementary ones, collected in b_c. Expressing (46) in terms of these values gives

$$
\mathcal{Z}b + \mathcal{Z}_c b_c = \begin{pmatrix} 0 \\ 0 \\ \cdots \\ 0 \\ \det(M + NE_{1,\sqrt{\lambda}}(1)) \end{pmatrix} \tag{50}
$$

where $\mathcal{Z}$ and $\mathcal{Z}_c$ are $(2r \times 2r)$ matrices built from the various components of M and N. To state b, b_c, $\mathcal{Z}$ and $\mathcal{Z}_c$ explicitly, we need to introduce several indices which refer to the ways in which the boundary data are re-distributed within each of the four boundary data groups. Let i, j, k, l be indices: all of which can take on values from 0 to r and such that $i + j + k + l = 2r$. Let $\{a_1, \ldots, a_r\}$ and $\{c_1, \ldots, c_r\}$ be arbitrary permutations of the numbers $\{1, \ldots, r\}$, and also let $\{b_1, \ldots, b_r\}$ and $\{d_1, \ldots, d_r\}$ be permutations of the numbers $\{r + 1, \ldots, 2r\}$. These index groups are such that m_{a_i} acts on boundary data in $u_{1,\sqrt{\lambda}}(0)$, m_{b_j} acts in $v_{1,\sqrt{\lambda}}(0)$, n_{c_k} acts in $u_{1,\sqrt{\lambda}}(1)$ and finally n_{d_l} acts in $v_{1,\sqrt{\lambda}}(1)$. The general form of b, b_c, $\mathcal{Z}$ and $\mathcal{Z}_c$ are discussed in appendix C, from which it is clear that b can be expressed through b_c only if the matrix $\mathcal{Z}$ is invertible. The choice of the data b has to guarantee this is indeed the case. That this is always possible follows from the fact that M and N define boundary conditions such that (46) has a unique solution for λ an eigenvalue. If a suitable choice of b were not possible, the boundary value problem would not have a unique solution.

For the example described by (43), the most natural choice for b, b_c, is

$$
b = \begin{pmatrix} u_{1,\sqrt{\lambda},1}(l/2) \\ u_{1,\sqrt{\lambda},2}(l/2) \\ v_{1,\sqrt{\lambda},1}(l/2) \\ v_{1,\sqrt{\lambda},2}(l/2) \end{pmatrix} \qquad b_c = \begin{pmatrix} u_{1,\sqrt{\lambda},1}(-l/2) \\ u_{1,\sqrt{\lambda},2}(-l/2) \\ v_{1,\sqrt{\lambda},1}(-l/2) \\ v_{1,\sqrt{\lambda},2}(-l/2) \end{pmatrix} \tag{51}
$$

so that $\mathcal{Z} = N \ (= I_4)$ and $\mathcal{Z}_c = M \ (= -\mathrm{diag}(\mathrm{e}^{\mathrm{i}\mu l}, \mathrm{e}^{-\mathrm{i}\mu l}, \mathrm{e}^{\mathrm{i}\mu l}, \mathrm{e}^{-\mathrm{i}\mu l}))$. This guarantees $\mathcal{Z}$ is invertible and b can be expressed through b_c.

Alternatively one could, for instance, choose

$$b^{(\text{alt})} = \begin{pmatrix} u_{1,\sqrt{\lambda},1}(-l/2) \\ u_{1,\sqrt{\lambda},2}(-l/2) \\ v_{1,\sqrt{\lambda},1}(-l/2) \\ v_{1,\sqrt{\lambda},2}(-l/2) \end{pmatrix} \qquad b_c^{(\text{alt})} = \begin{pmatrix} u_{1,\sqrt{\lambda},1}(l/2) \\ u_{1,\sqrt{\lambda},2}(l/2) \\ v_{1,\sqrt{\lambda},1}(l/2) \\ v_{1,\sqrt{\lambda},2}(l/2) \end{pmatrix}. \tag{52}$$

In this case

$$\mathcal{Z}^{(\text{alt})} = M \qquad \mathcal{Z}_c^{(\text{alt})} = N.$$

Again, $\mathcal{Z}$ is invertible and b can be expressed through b_c. Clearly, there are many other choices of b, b_c and the associated $\mathcal{Z}$, $\mathcal{Z}_c$.

Going back to the general formalism, given a suitable particular choice of $\mathcal{Z}$, this allows us to express the $2r$ data b by the complementary $2r$ data b_c. The explicit expression is given by equation (C.3) in appendix C. The entries of b can now be substituted into the left-hand side of (33) and the terms collected together. As discussed in appendix C this leads to (44) with

$$\mathcal{B}^{-1} = \sum_{\alpha=1}^{k} \mathcal{Z}^{-1}_{(i+j+\alpha)(2r)} v_{1,0,c_\alpha}(1)^* - \sum_{\alpha=1}^{l} \mathcal{Z}^{-1}_{(i+j+k+\alpha)(2r)} u_{1,0,d_\alpha-r}(1)^*$$

$$- \sum_{\alpha=1}^{i} \mathcal{Z}^{-1}_{\alpha(2r)} v_{1,0,a_\alpha}(0)^* + \sum_{\alpha=1}^{j} \mathcal{Z}^{-1}_{(i+\alpha)(2r)} u_{1,0,b_\alpha-r}(0)^* \tag{53}$$

where $\mathcal{Z}^{-1}_{\beta\gamma}$ refers to the $(\beta\gamma)$-component of $\mathcal{Z}^{-1}$.

To illustrate the use of this result let us apply it again to the example (43).
For the choice (51) we have $i = 0$, $j = 0$, $k = 2$, $l = 2$ and we obtain

$$[u_{1,\sqrt{\lambda}}(x)v_{1,0}(x)^* - u_{1,0}(x)^* v_{1,\sqrt{\lambda}}(x)]^{l/2}_{-l/2} = -\det(M + NE_{1,\sqrt{\lambda}}(l/2))u_{1,0,2}(l/2)^*.$$

For the choice (52) we have $i = 2$, $j = 2$, $k = 0$, $l = 0$ and we obtain

$$[u_{1,\sqrt{\lambda}}(x)v_{1,0}(x)^* - u_{1,0}(x)^* v_{1,\sqrt{\lambda}}(x)]^{l/2}_{-l/2} = -\det(M + NE_{1,\sqrt{\lambda}}(l/2))\,e^{i\mu l} u_{1,0,2}(-l/2)^*.$$

Comparing with (44) we see that for the choice (51) $\mathcal{B}^{-1} = -u_{1,0,2}(l/2)^*$ and for the choice (52) $\mathcal{B}^{-1} = -e^{i\mu l} u_{1,0,2}(-l/2)^*$. Taking into account the boundary conditions for the zero mode $u_{1,0}(x)$, the two answers are seen to agree. Furthermore, this answer agrees with the result calculated in [28].

Returning to the general case, we see that we have proved the result (45) with $\mathcal{B}$ given by (53). The proof now proceeds as in section 3 and we once again find the result (39). The function $y_1(x)$ in this result is the zero mode, and satisfies the boundary conditions, but it has to be appropriately normalized:

$$y_1(x) = \sum_{\sigma=1}^{2r} \text{adj}(M + NE_{1,0}(1))_{\sigma 2r} y_1^{(\sigma)}(x) \tag{54}$$

where the $y_1^{(\sigma)}(x)$ are the $2r$ fundamental solutions chosen to satisfy $Y_1(0) = I_{2r}$.

5. Conclusion

The two main results of this paper are (21) and (39). They give expressions for the ratio of functional determinants in terms of the nature of the boundary conditions and the solution of the homogeneous equations formed from the operators in question. The first result holds if

the equations have no zero modes, and the second holds if such an eigenvalue exists, but has been excluded from the evaluation of the functional determinant. These results agree with those obtained in a previous paper [1], but now the range of operators for which they are valid have been considerably extended to: those which have a metric $P_j(x) \neq 1$, those with negative eigenvalues and systems of operators. Although the results are simple to state, a slightly more thorough appreciation of the method is required in order to apply them to a particular case. For instance, the solution $y_1(x)$, which appears in the results, is the solution of the homogeneous equation satisfying the boundary conditions. This solution is only defined up to a constant, but a particular choice for this constant has to be made if the simpler form of (21)—given by (31)—or the zero-mode result (39), is to be used. An explicit form for $y_1(x)$ is given by (54). The choice of normalization originates from requiring that the right-hand side of (29) or (46) is the required determinant, but with a constant of proportionality which is equal to 1.

Once the suitably normalized solution $y_1(x)$ has been obtained, the rest of the calculation is straightforward. The result only depends on this function—and on none of the other eigenfunctions—and on the matrices M and N which define the boundary conditions of the problem under consideration. In addition, in some applications, the norm $\langle y_1 | y_1 \rangle$ will cancel out with the Jacobian of the transformation to collective coordinates, and therefore will not be required. In this case, however, it will be necessary to check that the zero-mode has the same normalization as has been adopted in the derivation of (39). So for this case only the properties of $y_1(x)$ at the boundaries would be required. If an analytic expression for $y_1(x)$ cannot be obtained, there should be little difficulty in obtaining numerical values for the boundary data on this function. While the proof of the results which we have obtained are easily accessible, they need not be understood in order to apply the results to a particular problem.

We believe that the results presented here cover a wide range of problems where they are likely to prove useful. There are still a number of possible extensions that are open to investigation. Examples include operators with derivatives higher than the second, single determinants rather than ratios of determinants and operators in more than one dimension. We hope that, in addition to the concrete results which we have obtained, this paper will serve to stimulate work on these and related problems.

Acknowledgments

Appendix A. Self-adjoint condition and related questions

In this appendix we will consider two technical points encountered in section 3. They are

(1) *Condition for problem to be self-adjoint.* The boundary conditions considered in section 3 are given by (22) and (23):

$$
\begin{aligned}
m_{11}u_{j,\sqrt{\lambda}}(0) + m_{12}v_{j,\sqrt{\lambda}}(0) + n_{11}u_{j,\sqrt{\lambda}}(1) + n_{12}v_{j,\sqrt{\lambda}}(1) = 0 \\
m_{21}u_{j,\sqrt{\lambda}}(0) + m_{22}v_{j,\sqrt{\lambda}}(0) + n_{21}u_{j,\sqrt{\lambda}}(1) + n_{22}v_{j,\sqrt{\lambda}}(1) = 0.
\end{aligned}
\tag{A.1}
$$

Suppose that $U^{(\mathrm{I})}_{j,\sqrt{\lambda}}(x)$ and $U^{(\mathrm{II})}_{j,\sqrt{\lambda}}(x)$ are any two functions (which are not, in general, solutions of (2)) which satisfy these boundary conditions. The condition for the problem to be self-adjoint is that

$$
\left[U^{(\mathrm{I})}_{j,\sqrt{\lambda}}(x) V^{(\mathrm{II})}_{j,\sqrt{\lambda}}(x)^* - U^{(\mathrm{II})}_{j,\sqrt{\lambda}}(x)^* V^{(\mathrm{I})}_{j,\sqrt{\lambda}}(x) \right]_0^1 = 0
\tag{A.2}
$$

where, as in the main text, $V_{j,\sqrt{\lambda}}(x) = P_j(x)U'_{j,\sqrt{\lambda}}(x)$. We wish to solve for any two members of the set $\left\{U^{(I)}_{j,\sqrt{\lambda}}(0), V^{(I)}_{j,\sqrt{\lambda}}(0), U^{(I)}_{j,\sqrt{\lambda}}(1), V^{(I)}_{j,\sqrt{\lambda}}(1)\right\}$ in terms of the other two by using the two boundary conditions (and similarly for the second solution II). Substituting these four functions into (A.2) in terms of the other four will give us the conditions that need to be imposed on the matrices M and N for the problem to be self-adjoint.

(2) *Proof of the first equality in equation (34).* In this case the $\lambda = 0$ solution $u_{1,0}(x)$ will satisfy the boundary conditions (A.1), but the $\lambda \neq 0$ solution will only satisfy one boundary condition and a normalization condition, that is (see equation (29)):

$$m_{11}u_{1,\sqrt{\lambda}}(0) + m_{12}v_{1,\sqrt{\lambda}}(0) + n_{11}u_{1,\sqrt{\lambda}}(1) + n_{12}v_{1,\sqrt{\lambda}}(1) = 0$$
$$m_{21}u_{1,\sqrt{\lambda}}(0) + m_{22}v_{1,\sqrt{\lambda}}(0) + n_{21}u_{1,\sqrt{\lambda}}(1) + n_{22}v_{1,\sqrt{\lambda}}(1) = \det(M + NE_{1,\sqrt{\lambda}}(1)). \tag{A.3}$$

As discussed in section 3, we wish to solve for any two members of the set $\{u_{1,\sqrt{\lambda}}(0), v_{1,\sqrt{\lambda}}(0), u_{1,\sqrt{\lambda}}(1), v_{1,\sqrt{\lambda}}(1)\}$ in terms of the other two and $\det(M + NE_{1,\sqrt{\lambda}}(1))$ by using the conditions (A.3). Similarly, we wish to solve for any two members of the set $\{u_{1,0}(0)^*, v_{1,0}(0)^*, u_{1,0}(1)^*, v_{1,0}(1)^*\}$, in terms of the other two, this time using (A.1). Substituting these four functions into the left-hand side of (33), in terms of the other four, will enable us to show that

$$[u_{1,\sqrt{\lambda}}(x)v_{1,0}(x)^* - u_{1,0}(x)^*v_{1,\sqrt{\lambda}}(x)]_0^1 \propto \det(M + NE_{1,\sqrt{\lambda}}(1)) \tag{A.4}$$

where the constant of proportionality is independent of λ, and only depends on the nature of the boundary conditions and on the $\lambda = 0$ solution $\mathbf{u}_{1,0}(x)$ at the boundaries.

It is clear that these two questions are related. In fact, the proof of the first point is a special case of the proof of the second; we simply need to set $\det(M + NE_{1,\sqrt{\lambda}}(1))$ equal to zero everywhere. We will prove the result first in the case of separated boundary conditions and then for non-separated ones.

(i) For separated boundary conditions we have $\det N = 0$ and $\det M = 0$, and M and N may be chosen to have the form (14).

If $m_{12} \neq 0$ and $n_{22} \neq 0$, then (A.3) may be written in the form

$$v_{1,\sqrt{\lambda}}(0) = -\frac{m_{11}}{m_{12}}u_{1,\sqrt{\lambda}}(0)$$
$$v_{1,\sqrt{\lambda}}(1) = \frac{\det(M + NE_{1,\sqrt{\lambda}}(1))}{n_{22}} - \frac{n_{21}}{n_{22}}u_{1,\sqrt{\lambda}}(1).$$

Equivalent results hold when $\lambda = 0$ if the determinant is set equal to zero. Eliminating the v in terms of the u yields

$$[u_{1,\sqrt{\lambda}}(x)v_{1,0}(x)^* - u_{1,0}(x)^*v_{1,\sqrt{\lambda}}(x)]_0^1 = -\frac{u_{1,0}(1)^*}{n_{22}}\det(M + NE_{1,\sqrt{\lambda}}(1))$$
$$+ u_{1,0}(1)^*u_{1,\sqrt{\lambda}}(1)\left[\frac{n_{21}}{n_{22}} - \frac{n_{21}^*}{n_{22}^*}\right] - u_{1,0}(0)^*u_{1,\sqrt{\lambda}}(0)\left[\frac{m_{11}}{m_{12}} - \frac{m_{11}^*}{m_{12}^*}\right].$$

If we first of all assume that all of the boundary conditions are satisfied, then the determinant is not present and we see that the general condition for the operator to be self-adjoint is that the ratios m_{11}/m_{12} and n_{21}/n_{22} be real. Since we always have the freedom to multiply the first line of (11) by an arbitrary complex number and the second line by another arbitrary complex number, we can always choose m_{12} and n_{22} to be real, in which case we deduce that m_{11} and n_{21} should also be real. Therefore if the operator is self-adjoint, the matrices M and N can always be chosen to be real.

If $m_{12} = 0$, $n_{22} \neq 0$, then $u_{1,\sqrt{\lambda}}(0) = 0$ and $u_{1,0}(0) = 0$. The above expression then tells us only that n_{21}/n_{22} must be real if the operator is to be self-adjoint. But now m_{11} and n_{22} may be chosen to be real, and we once again find that M and N may be taken to be real. The remaining cases where $n_{22} = 0$ may be treated in the same way.

In summary, when $\det M = \det N = 0$, if the operator is self-adjoint then M and N may always be chosen to be real, and

$$[u_{1,\sqrt{\lambda}}(x)v_{1,0}(x)^* - u_{1,0}(x)^*v_{1,\sqrt{\lambda}}(x)]_0^1$$
$$= \begin{cases} -\frac{u_{1,0}(1)^*}{n_{22}} \det(M + NE_{1,\sqrt{\lambda}}(1)) & \text{if } n_{22} \neq 0 \\ +\frac{v_{1,0}(1)^*}{n_{21}} \det(M + NE_{1,\sqrt{\lambda}}(1)) & \text{if } n_{21} \neq 0. \end{cases} \quad \text{(A.5)}$$

Through the boundary condition for the zero mode, these two forms are clearly equivalent if both n_{21} and n_{22} are non-zero.

(ii) Suppose that $\det N \neq 0$. Then multiplying (29) by N^{-1} and taking $j = 1$ gives

$$\begin{pmatrix} u_{1,\sqrt{\lambda}}(1) \\ v_{1,\sqrt{\lambda}}(1) \end{pmatrix} = \frac{1}{\det N} \begin{pmatrix} n_{22} & -n_{12} \\ -n_{21} & n_{11} \end{pmatrix} \begin{pmatrix} 0 \\ \det(M + NE_{1,\sqrt{\lambda}}(1)) \end{pmatrix} - \begin{pmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{pmatrix} \begin{pmatrix} u_{1,\sqrt{\lambda}}(0) \\ v_{1,\sqrt{\lambda}}(0) \end{pmatrix}$$

where the d_{ij} are the elements of the matrix $D \equiv N^{-1}M$. Substituting for $u_{1,\sqrt{\lambda}}(1)$ and $v_{1,\sqrt{\lambda}}(1)$ (and their $\lambda = 0$ counterparts, which do not contain the $\det(M + NE_{1,\sqrt{\lambda}}(1))$ term) gives

$$[u_{1,\sqrt{\lambda}}(x)v_{1,0}(x)^* - u_{1,0}(x)^*v_{1,\sqrt{\lambda}}(x)]_0^1$$
$$= -\frac{(n_{11}u_{1,0}(1)^* + n_{12}v_{1,0}(1)^*)}{\det N} \det(M + NE_{1,\sqrt{\lambda}}(1))$$
$$+ u_{1,0}(0)^*v_{1,\sqrt{\lambda}}(0)\{1 + d_{12}d_{21}^* - d_{11}^*d_{22}\}$$
$$+ v_{1,0}(0)^*v_{1,\sqrt{\lambda}}(0)\{d_{12}d_{22}^* - d_{12}^*d_{22}\}$$
$$- v_{1,0}(0)^*u_{1,\sqrt{\lambda}}(0)\{1 + d_{12}^*d_{21} - d_{11}d_{22}^*\}$$
$$- u_{1,0}(0)^*u_{1,\sqrt{\lambda}}(0)\{d_{21}d_{11}^* - d_{21}^*d_{11}\}.$$

If all boundary conditions were satisfied, the first term on the right-hand side of this expression would be absent, and the general conditions for the operator to be self-adjoint are given by the vanishing of the brackets in the four remaining terms

$$d_{11}^*d_{22} - d_{12}d_{21}^* = 1 \qquad d_{11}d_{21}^* = d_{11}^*d_{21}$$
$$d_{11}d_{22}^* - d_{12}^*d_{21} = 1 \qquad d_{12}d_{22}^* = d_{12}^*d_{22}. \quad \text{(A.6)}$$

An examination of the conditions (A.6) shows that they may be written in the alternative form

$$d_{ij} = r_{ij}\,e^{i\alpha} \qquad r_{11}r_{22} - r_{12}r_{21} = 1 \qquad 0 \leqslant \alpha < 2\pi \quad r_{ij} \in \mathbb{R}. \quad \text{(A.7)}$$

In other words, $D = R\,e^{i\alpha}$, where R is a 2×2 real matrix with entries r_{ij} and $\det R = 1$. Therefore if we multiply (11) by N^{-1} we see that we may take $M = R\,e^{i\alpha}$, $N = I_2$, or equivalently if we multiply by a real non-singular matrix N_R, $M = M_R\,e^{i\alpha}$, $N = N_R$, where $M_R = N_R R$. Note that $\det M_R = \det N_R$.

Appendix B. Asymptotic behaviour of solutions at endpoints

In this appendix we are going to analyse the behaviour of $(d/d\lambda) \ln \det[M + NE_{j,\sqrt{\lambda}}(1)]$ for $\text{Im}\,\sqrt{\lambda} \to \pm\infty$ as it is needed in equation (16). We will omit the index j and consider the

general Sturm–Liouville problem

$$L = -\frac{d}{dx}\left(P(x)\frac{d}{dx}\right) + R(x)$$

with the boundary conditions (11) imposed. The zeta function associated with this problem is then given by equation (15),

$$\zeta_L(s) = \frac{1}{2\pi i}\int_\gamma d\lambda\,\lambda^{-s}\frac{d}{d\lambda}\ln\det[M + NE_{\sqrt{\lambda}}(1)].$$

The meromorphic structure of the zeta function is determined by the large-Im $\sqrt{\lambda}$ behaviour of the integrand. The results in [29] suggest that as Im $\sqrt{\lambda} \to \pm\infty$ the asymptotic expansion has the general form

$$\frac{d}{d\lambda}\ln\det[M + NE_{\sqrt{\lambda}}(1)] \sim \sum_{n=1}^\infty (\sqrt{\lambda})^{-n}A_{n-1}. \tag{B.1}$$

Note that exponentially small terms have been dropped.

We will now show that the coefficients A_{n-1} are related to the associated heat kernel coefficients and use this correspondence to prove that the first two coefficients do not depend on $R(x)$. As a consequence we can trivially conclude the behaviour (17).

We start summarizing some well-known facts about the heat kernel coefficients and their relationship to the zeta function [30, 26]. The heat trace is defined as

$$K(t) = \sum_l e^{-\lambda_l t}$$

where λ_l are the eigenvalues of the operator under consideration. As $t \to 0$, this sum clearly diverges, since we are summing over infinitely many eigenvalues. The behaviour as $t \to 0$ may be extracted from a classical theorem of Weyl [31], which, in the present context, states that for a second-order elliptic differential operator the eigenvalues behave asymptotically for $l \to \infty$ as

$$\lambda_l^{1/2} \sim \frac{\pi l}{\int_0^1 dx\,\frac{1}{\sqrt{P(x)}}}.$$

With the help of a resummation,

$$\sum_{l=-\infty}^\infty e^{-tl^2} = \sqrt{\frac{\pi}{t}}\sum_{l=-\infty}^\infty \exp\left(-\frac{\pi^2 l^2}{t}\right)$$

it is seen that this implies $K(t) = \mathcal{O}(t^{-1/2})$. In more detail, one can show the asymptotic $t \to 0$ behaviour

$$K(t) \sim \sum_{j=0}^\infty a_j t^{(j-1)/2} \tag{B.2}$$

where exponentially small terms as $t \to 0$ have been neglected. Here, a_l are the so-called heat kernel coefficients. They depend on $P(x)$, $R(x)$ and on the boundary conditions imposed. We have, for example,

$$a_0 = (4\pi)^{-1/2}\int_0^1 dx\,\frac{1}{\sqrt{P(x)}} \qquad a_1 = c(M, N)$$

where the constant $c(M, N)$, as indicated, depends on the boundary condition imposed. The next coefficient a_2 involves the dependence on $R(x)$. As this is of no relevance for us, we do not display higher coefficients.

The heat kernel coefficients determine the residues and certain function values of the zeta function. To show how the relationship is derived we assume that no zero modes are present; otherwise, in the following calculations we have to exclude them explicitly.

First by definition

$$\zeta_L(s) = \sum_{l=0}^{\infty} \lambda_l^{-s} = \frac{1}{\Gamma(s)} \sum_{l=0}^{\infty} \int_0^{\infty} dt\, t^{s-1}\, e^{-\lambda_l t} = \frac{1}{\Gamma(s)} \int_0^{\infty} dt\, t^{s-1} K(t)$$

valid for $\mathrm{Re}\, s > 1/2$. As is clear, the meromorphic structure of $\zeta_L(s)$ is related to the $t \to 0$ behaviour of $K(t)$. Thus the poles of $\zeta_L(s)\Gamma(s)$ are determined by the integrals

$$\int_0^1 dt\, t^{s-1} \sum_{j=0}^{\infty} a_j t^{(j-1)/2}.$$

In detail we have

$$\mathrm{Res}\, \zeta_L(z) = \frac{a_{1-2z}}{\Gamma(z)} \qquad \text{for } z = \frac{1}{2}, -\frac{2l+1}{2}, l \in \mathbb{N}_0$$

$$\zeta_L(-q) = (-1)^q q!\, a_{1+2q} \qquad \text{for } q \in \mathbb{N}_0. \tag{B.3}$$

The asymptotic expansion (B.1) determines the above properties of the zeta function and thus relates A_n and a_n. Proceeding as before, see (18), we shrink the contour to the branch cut at the angle θ. For the case without zero modes, as $\lambda \to 0$ we have the behaviour λ^{-s} and as $\lambda \to \infty$ we have $\lambda^{-s-1/2}$. The $\lambda \to 0$ behaviour imposes $\mathrm{Re}\, s < 1$, whereas the $\lambda \to \infty$ behaviour imposes $\mathrm{Re}\, s > 1/2$. This shows that the representation, as given, is valid for $1/2 < \mathrm{Re}\, s < 1$. It also shows that the residues and function values, (B.3), which all lie to the left of $\mathrm{Re}\, s > 1/2$, are solely determined by the large-λ behaviour. Keeping only the relevant terms to reproduce (B.3), we continue

$$\zeta_L(s) \sim e^{is(\pi-\theta)} \frac{\sin(\pi s)}{\pi} \int_1^{\infty} d\lambda\, \lambda^{-s}\, e^{i\theta} \sum_{n=1}^{\infty} e^{-in\theta/2} \lambda^{-n/2} A_{n-1}$$

$$= e^{is(\pi-\theta)} \frac{\sin(\pi s)}{\pi} \sum_{n=1}^{\infty} e^{i\theta(1-n/2)} \frac{A_{n-1}}{s-1+n/2} \tag{B.4}$$

which can be analysed easily in the whole complex plane. We see that for n odd,

$$\mathrm{Res}\, \zeta_L\left(1 - \frac{n}{2}\right) = \frac{i}{\pi} A_{n-1}$$

which shows

$$A_{n-1} = -i\pi \frac{a_{n-1}}{\Gamma(1-n/2)} \tag{B.5}$$

whereas for n even,

$$\zeta_L\left(1 - \frac{n}{2}\right) = A_{n-1}$$

and so

$$A_{n-1} = (-1)^{n/2-1}(n/2-1)!\, a_{n-1}. \tag{B.6}$$

In particular,

$$A_0 = -\frac{i\pi a_0}{\Gamma(1/2)} = -\frac{i}{2} \int_0^1 dx \frac{1}{\sqrt{P(x)}} \qquad A_1 = a_1 = c(M, N)$$
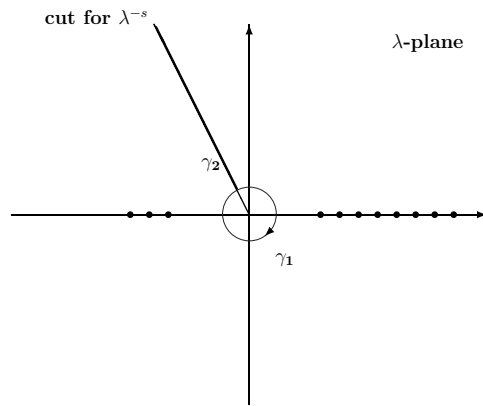
and (17) follows.

Figure 2. Contour γ in the complex plane.

Note that in (B.4) we have $e^{i\theta/2}\sqrt{\lambda}$ with positive imaginary part. A negative imaginary part, such as in $-e^{i\theta/2}\sqrt{\lambda}$, changes the sign of A_{n-1} for n odd.

If there are zero modes, say r in number, the equation for a_1 changes slightly. First, given we exclude the zero mode from the definition of the zeta function, we have now

$$\zeta_L(s) = \frac{1}{\Gamma(s)} \int_0^\infty \mathrm{d}t \, t^{s-1}(K(t) - r) \qquad \mathrm{Re}\, s > \frac{1}{2}.$$

This shows that (B.3) remains unchanged apart from

$$\zeta_L(0) = a_1 - r.$$

Given there are r zero modes, for $|\lambda| \ll 1$, we have

$$\frac{\mathrm{d}}{\mathrm{d}\lambda} \ln \det[M + N E_{\sqrt{\lambda}}(1)] \sim \frac{r}{\lambda} + \cdots.$$

Repeating the discussion below (B.3), we see that this time the $\lambda \to 0$ behaviour imposes $\mathrm{Re}\, s < 0$, which contradicts the condition $\mathrm{Re}\, s > 1/2$ from $|\lambda| \to \infty$. Therefore, we cannot shrink the contour to the cut, but instead use the contour given in figure 2, consisting of a small circle γ_1 of radius ϵ, and of γ_2 being the part of γ shrunk to the cut.

Along the contour γ_2 we can proceed as previously and obtain

$$\zeta_{L,\gamma_2}(s) = e^{is(\pi-\theta)}\frac{\sin(\pi s)}{\pi} \int_\epsilon^\infty \mathrm{d}\lambda \, \lambda^{-s} \frac{\mathrm{d}}{\mathrm{d}\lambda} \ln \det[M + N E_{e^{i\theta/2}\sqrt{\lambda}}(1)]. \qquad (B.7)$$

For the contributions along the circle γ_1 we obtain

$$\zeta_{L,\gamma_1}(s) = -e^{is(\pi-\theta)}\frac{\sin(\pi s)}{\pi}\frac{r\epsilon^{-s}}{s}. \qquad (B.8)$$

The contribution of (B.7) to the quantities in (B.3) is evaluated precisely as before. In addition, as $s \to 0$, (B.8) produces

$$\zeta_{L,\gamma_1}(0) = -r.$$

As a result, the asymptotic behaviour is determined again through equations (B.5) and (B.6).

Appendix C. Zero modes in systems of differential operators

In section 4 we introduced the vectors b and b_c and the matrices $\mathcal{Z}$ and $\mathcal{Z}_c$. The $2r$-dimensional vectors b and b_c together contain the $4r$ boundary data: r data coming from each of $u_{1,\sqrt{\lambda}}(0)$, $v_{1,\sqrt{\lambda}}(0)$, $u_{1,\sqrt{\lambda}}(1)$ and $v_{1,\sqrt{\lambda}}(1)$. The $2r \times 2r$ matrices $\mathcal{Z}$ and $\mathcal{Z}_c$ together contain all the elements of the matrices M and N but rearranged in a way which corresponds to the organization of the boundary data in b and b_c. The purpose of this appendix is to make explicit the notation required to describe which of the $2r$ boundary data goes into b and which goes into b_c and which of the elements of M and N go into $\mathcal{Z}$ and which go into $\mathcal{Z}_c$.

To do this we introduce indices i, j, k, l and permutations $\{a_1, \ldots, a_r\}$, $\{b_1, \ldots, b_r\}$, $\{c_1, \ldots, c_r\}$ and $\{d_1, \ldots, d_r\}$ as in section 4. Let us recall that these index groups are such that m_{a_i} acts on boundary data in $u_{1,\sqrt{\lambda}}(0)$, m_{b_j} acts in $v_{1,\sqrt{\lambda}}(0)$, n_{c_k} acts in $u_{1,\sqrt{\lambda}}(1)$ and n_{d_l} acts in $v_{1,\sqrt{\lambda}}(1)$. So if

$$
b = \begin{pmatrix}
u_{1,\sqrt{\lambda},a_1}(0) \\
\cdots \\
u_{1,\sqrt{\lambda},a_i}(0) \\
v_{1,\sqrt{\lambda},b_1-r}(0) \\
\cdots \\
v_{1,\sqrt{\lambda},b_j-r}(0) \\
u_{1,\sqrt{\lambda},c_1}(1) \\
\cdots \\
u_{1,\sqrt{\lambda},c_k}(1) \\
v_{1,\sqrt{\lambda},d_1-r}(1) \\
\cdots \\
v_{1,\sqrt{\lambda},d_l-r}(1)
\end{pmatrix}
\qquad
b_c = \begin{pmatrix}
u_{1,\sqrt{\lambda},a_{i+1}}(0) \\
\cdots \\
u_{1,\sqrt{\lambda},a_r}(0) \\
v_{1,\sqrt{\lambda},b_{j+1}-r}(0) \\
\cdots \\
v_{1,\sqrt{\lambda},b_r-r}(0) \\
u_{1,\sqrt{\lambda},c_{k+1}}(1) \\
\cdots \\
u_{1,\sqrt{\lambda},c_r}(1) \\
v_{1,\sqrt{\lambda},d_{l+1}-r}(1) \\
\cdots \\
v_{1,\sqrt{\lambda},d_r-r}(1)
\end{pmatrix}
$$

then

$$
\mathcal{Z} = \begin{pmatrix}
m_{1a_1} & \cdots & m_{1a_i} & m_{1b_1} & \cdots & m_{1b_j} \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
m_{(2r)a_1} & \cdots & m_{(2r)a_i} & m_{(2r)b_1} & \cdots & m_{(2r)b_j} \\
n_{1c_1} & \cdots & n_{1c_k} & n_{1d_1} & \cdots & n_{1d_l} \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
n_{(2r)c_1} & \cdots & n_{(2r)c_k} & n_{(2r)d_1} & \cdots & n_{(2r)d_l}
\end{pmatrix}
\tag{C.1}
$$

and

$$
\mathcal{Z}_c = \begin{pmatrix}
m_{1a_{i+1}} & \cdots & m_{1a_r} & m_{1b_{j+1}} & \cdots & m_{1b_r} \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
m_{(2r)a_{i+1}} & \cdots & m_{(2r)a_r} & m_{(2r)b_{j+1}} & \cdots & m_{(2r)b_r} \\
n_{1c_{k+1}} & \cdots & n_{1c_r} & n_{1d_{l+1}} & \cdots & n_{1d_r} \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
n_{(2r)c_{k+1}} & \cdots & n_{(2r)c_r} & n_{(2r)d_{l+1}} & \cdots & n_{(2r)d_r}
\end{pmatrix}.
\tag{C.2}
$$

The notation is such that if one of the indices i, j, k, l equals zero, then the corresponding entries above are simply absent. It is clear from (50) that b can be expressed through b_c only

if the matrix $\mathcal{Z}$ is invertible. If this is the case then, for a particular choice of $\mathcal{Z}$, (50) allows us to express the $2r$ data b in terms of the complementary $2r$ data b_c as

$$
\begin{pmatrix}
u_{1,\sqrt{\lambda},a_1}(0) \\
\cdots \\
u_{1,\sqrt{\lambda},a_i}(0) \\
v_{1,\sqrt{\lambda},b_1-r}(0) \\
\cdots \\
v_{1,\sqrt{\lambda},b_j-r}(0) \\
u_{1,\sqrt{\lambda},c_1}(1) \\
\cdots \\
u_{1,\sqrt{\lambda},c_k}(1) \\
v_{1,\sqrt{\lambda},d_1-r}(1) \\
\cdots \\
v_{1,\sqrt{\lambda},d_l-r}(1)
\end{pmatrix}
= -\mathcal{Z}^{-1}\mathcal{Z}_c
\begin{pmatrix}
u_{1,\sqrt{\lambda},a_{i+1}}(0) \\
\cdots \\
u_{1,\sqrt{\lambda},a_r}(0) \\
v_{1,\sqrt{\lambda},b_{j+1}-r}(0) \\
\cdots \\
v_{1,\sqrt{\lambda},b_r-r}(0) \\
u_{1,\sqrt{\lambda},c_{k+1}}(1) \\
\cdots \\
u_{1,\sqrt{\lambda},c_r}(1) \\
v_{1,\sqrt{\lambda},d_{l+1}-r}(1) \\
\cdots \\
v_{1,\sqrt{\lambda},d_r-r}(1)
\end{pmatrix}
+ \mathcal{Z}^{-1}
\begin{pmatrix}
0 \\
\cdots \\
\cdots \\
\cdots \\
\cdots \\
\cdots \\
\cdots \\
\cdots \\
\cdots \\
\cdots \\
\cdots \\
0 \\
\det(M + NE_{1,\sqrt{\lambda}}(1))
\end{pmatrix}.
\tag{C.3}
$$

We may now substitute the $2r$ quantities on the left-hand side of (C.3) into the left-hand side of (33). As we have argued for the case $r = 1$, all terms that do not depend explicitly on the term containing $\det(M + NE_{1,\sqrt{\lambda}}(1))$ have to cancel each other due to the self-adjointness of the boundary value problem. Without attempting to state the conditions required for M and N (this does not do any harm simply because they are not needed), we keep only terms that do depend on the term $\det(M + NE_{1,\sqrt{\lambda}}(1))$, knowing the others *have* to cancel. In this way we arrive at

$$
[u_{1,\sqrt{\lambda}}(x)v_{1,0}(x)^* - u_{1,0}(x)^*v_{1,\sqrt{\lambda}}(x)]_0^1
$$

$$
= \det(M + NE_{1,\sqrt{\lambda}}(1))\left\{ \sum_{\alpha=1}^{k} \mathcal{Z}^{-1}_{(i+j+\alpha)(2r)} v_{1,0,c_\alpha}(1)^* \right.
$$

$$
- \sum_{\alpha=1}^{l} \mathcal{Z}^{-1}_{(i+j+k+\alpha)(2r)} u_{1,0,d_\alpha-r}(1)^* - \sum_{\alpha=1}^{i} \mathcal{Z}^{-1}_{\alpha(2r)} v_{1,0,a_\alpha}(0)^*
$$

$$
\left. + \sum_{\alpha=1}^{j} \mathcal{Z}^{-1}_{(i+\alpha)(2r)} u_{1,0,b_\alpha-r}(0)^* \right\}
\tag{C.4}
$$

where $\mathcal{Z}^{-1}_{\beta\gamma}$ refers to the $(\beta\gamma)$-component of $\mathcal{Z}^{-1}$. This is of the desired form (44) with $\mathcal{B}$ given by (53).

References

[1] Kirsten K and McKane A J 2003 *Ann. Phys.* **308** 502
[2] Gel'fand I M and Yaglom A M 1960 *J. Math. Phys.* **1** 48
[3] See, for example, Brézin E, Le Guillou J-C and Zinn-Justin J 1977 *Phys. Rev.* D **15** 1544
[4] Coleman S 1985 *Aspects of Symmetry: Selected lectures of Sidney Coleman* (Cambridge: Cambridge University Press)
[5] Levit S and Smilansky U 1977 *Proc. Am. Math. Soc.* **65** 299
[6] Dreyfuss T and Dym H 1978 *Duke Math. J.* **45** 15
[7] Forman R 1987 *Invent. Math.* **88** 447
[8] Forman R 1992 *Commun. Math. Phys.* **147** 485
[9] Burghelea D, Friedlander L and Kappeler T 1991 *Commun. Math. Phys.* **138** 1

[10] Burghelea D, Friedlander L and Kappeler T 1993 *Integral. Eqns. Oper. Theory* **16** 496
[11] Carron G 2002 *Am. J. Math.* **124** 307
[12] Burghelea D, Friedlander L and Kappeler T 1995 *Proc. Am. Math. Soc.* **123** 3027
[13] Saravi R E G, Muschietti M A and Solomin J E 1987 *Commun. Math. Phys.* **110** 641
[14] Saravi R E G, Rossini G L and Fuentes M 1992 *J. Phys. A: Math. Gen.* **25** 6743
[15] Barraza O A, Falomir H, Saravi R E G and Santangelo E M 1992 *J. Math. Phys.* **33** 2046
[16] Barraza O A 1994 *Commun. Math. Phys.* **163** 395
[17] Lesch M 1998 *Math. Nachr.* **194** 139
[18] Lesch M and Tolksdorf J 1998 *Commun. Math. Phys.* **193** 643
[19] Falomir H A, Saravi R E G, Muschietti M A, Santangelo E M and Solomin J E 1999 *Bull. Sci. Math.* **123** 233
[20] Rajaraman R 1982 *Solitons and Instantons* (Amsterdam: North-Holland)
[21] McKane A J and Tarlie M B 1995 *J. Phys. A: Math. Gen.* **28** 6931
[22] Kleinert H and Chervyakov A 1998 *Phys. Lett.* A **245** 345
[23] Kleinert H and Chervyakov A 1999 *J. Math. Phys.* **40** 6044
[24] Bordag M, Elizalde E and Kirsten K 1996 *J. Math. Phys.* **37** 895
[25] Bordag M, Kirsten K and Dowker J S 1996 *Commun. Math. Phys.* **182** 371
[26] Kirsten K 2001 *Spectral Functions in Mathematics and Physics* (London/Boca Raton, FL: Chapman and Hall/CRC Press)
[27] Tarlie M B, Shimshoni E and Goldbart P M 1994 *Phys. Rev.* B **49** 494
[28] Tarlie M B 1995 Nonequilibrium properties of mesoscopic superconducting rings *PhD Thesis* University of Illinois at Urbana-Champaign
[29] Olver F W J 1997 *Asymptotics and Special Functions* (Wellesley, MA: A K Peters) chapter 10.3
[30] Gilkey P B 1995 *Invariance Theory, the Heat Equation and the Atiyah–Singer Index Theorem* (Boca Raton, FL: CRC Press)
[31] Weyl H 1912 *Math. Ann.* **71** 441